Parameterized Complexity

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Seminar in Algorithms
186.182

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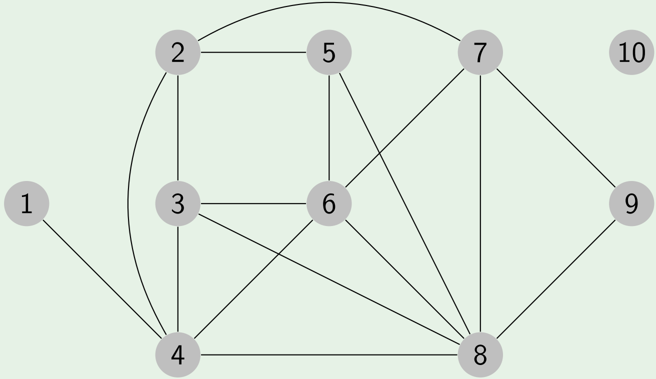
Motivation

Example



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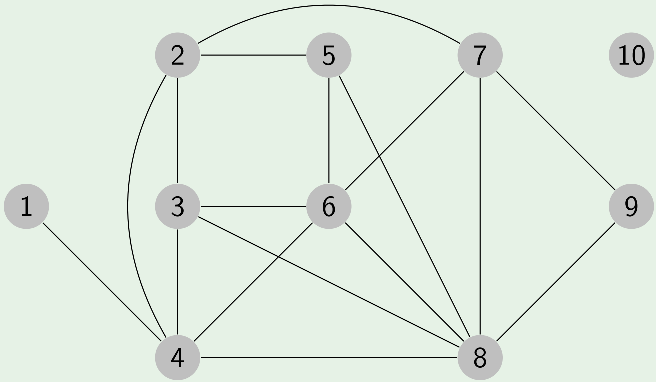

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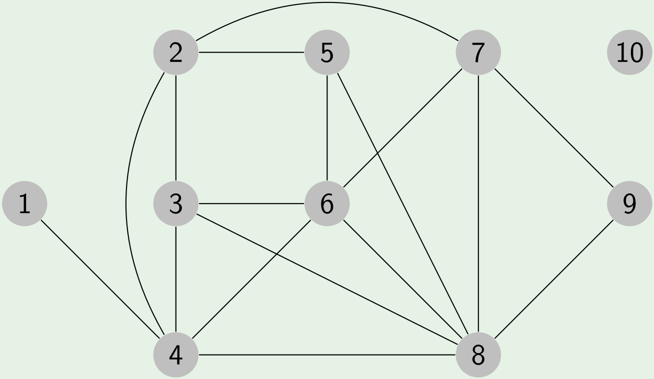

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- Is there a vertex cover of size at most 5?

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    - **Example:** VERTEX COVER, SAT, DOMINATING SET, . . .

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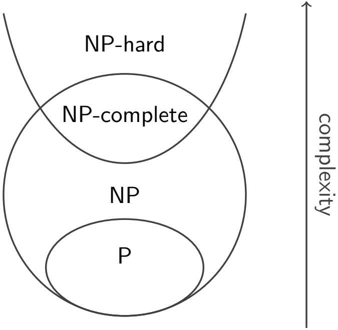

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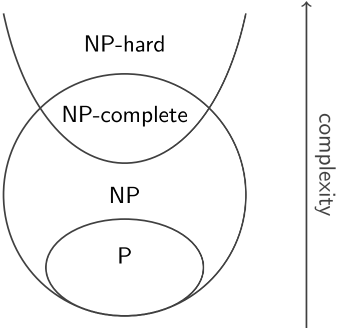

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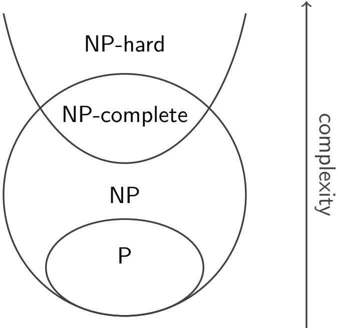

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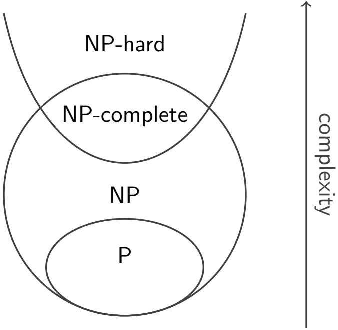

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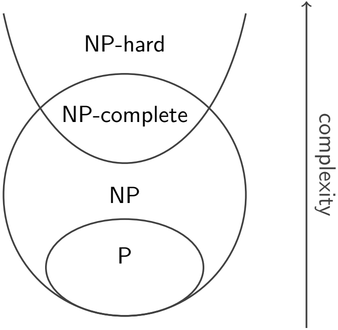

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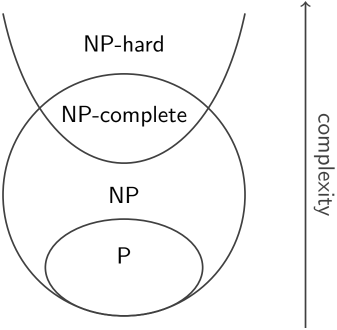

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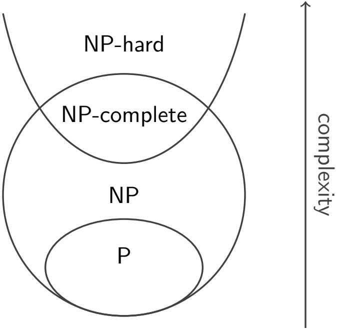

Conjecture

$$P \neq NP$$

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- Some parameters are useful, most are not!

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Parameterized complexity is based on a deal with the devil of intractability. *(*R.G. Downey and M.R. Fellows*)*

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    - *intractable* for $k \geq 3$

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The concept of FPT *belongs into the toolkit of every algorithm designer.*
*(*Rolf Niedermeier, Friedrich-Schiller-Universität Jena*)*

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Instance: *A bipartite graph* $G = (S, T, E)$ *with stations* S *and trains* T *and a positive integer* k.
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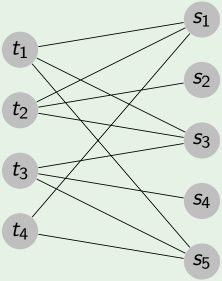

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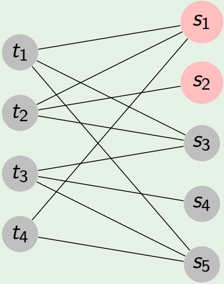

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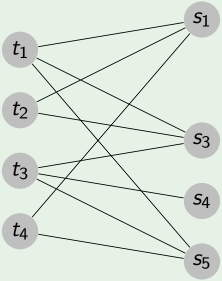

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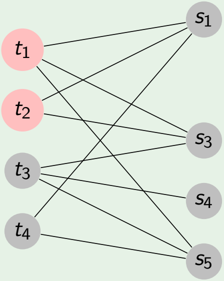

Example: Weihe's train problem

Definition (Weihe's reduction rules)

For $s, s' \in S$ and $t, t' \in T$. $N(v)$ denotes the of neighbours of v.

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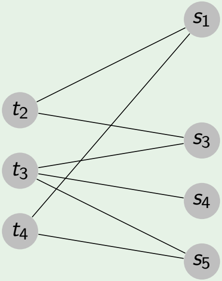

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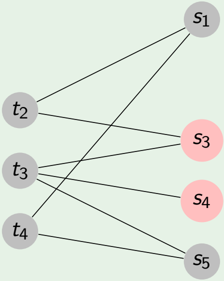

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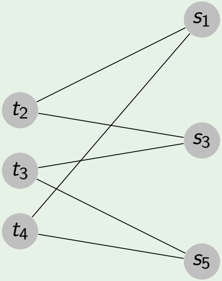

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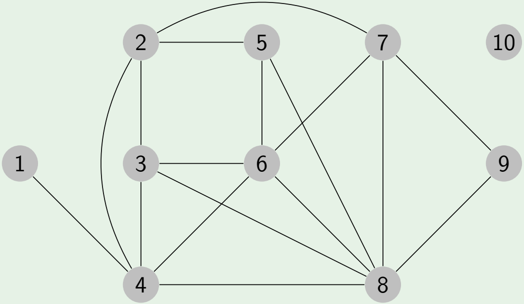
- VC3 For a vertex with degree $> k$, put this vertex into the cover and delete it form the graph.

Example: Buss's reduction

Example



Cover: $\{\}$ $k = 5$

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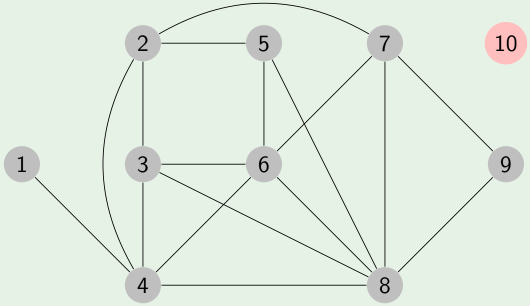

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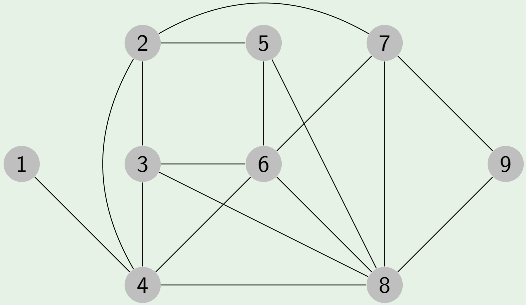

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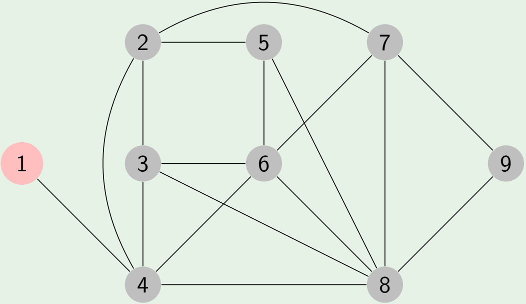

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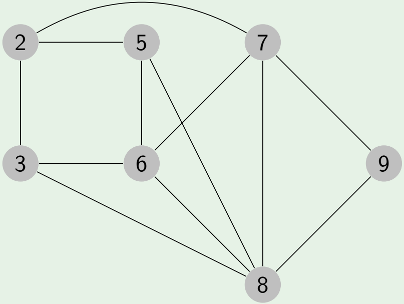

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Cover: $\{4\}$ $k = 4$

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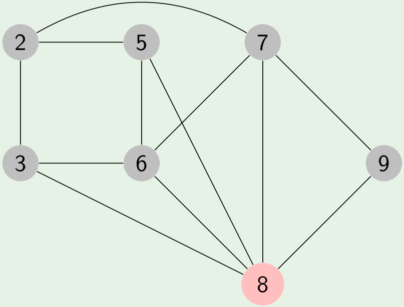

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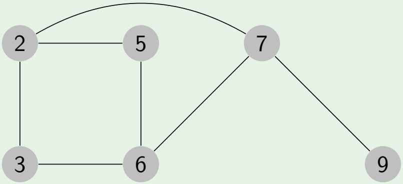

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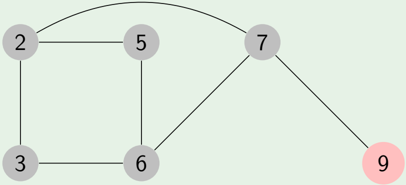

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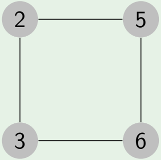

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Cover: $\{4, 8, 7\}$ $k = 2$

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Kernelizations can explain, and prove, why rules work so well in practice.
*(*Rolf Niedermeier, Friedrich-Schiller-Universität Jena*)*

Table of Contents

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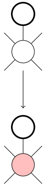

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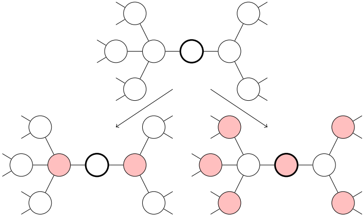

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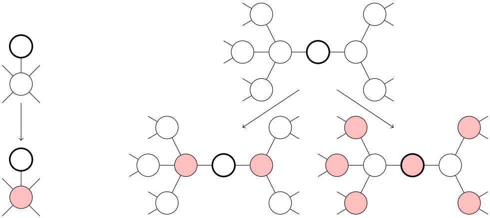

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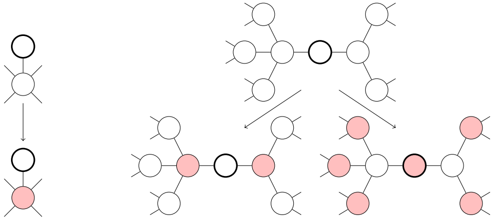

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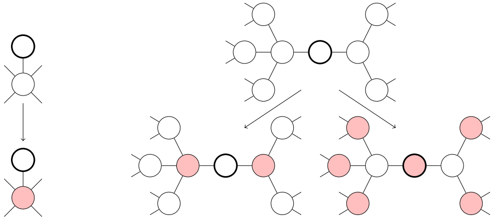

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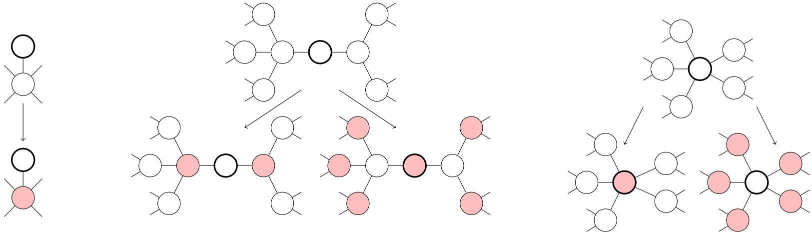

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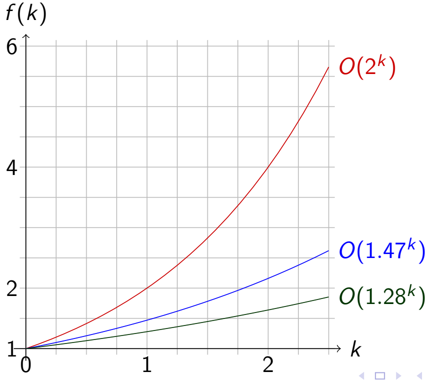

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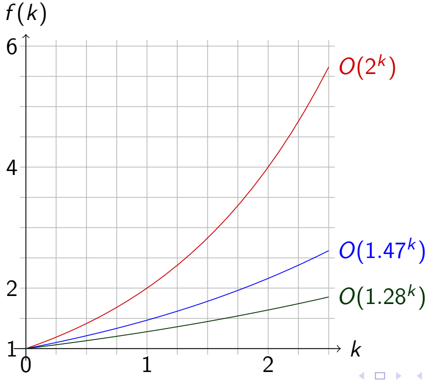

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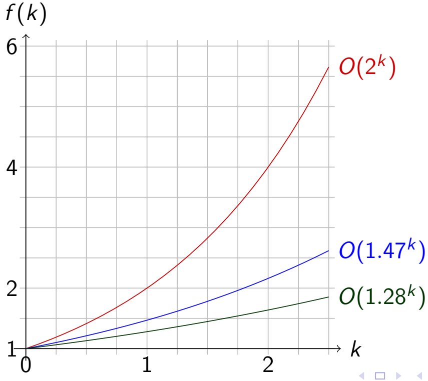

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> *The art of case distinction.*
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